

# INTERSECTIONS OF CERTAIN DELETED DIGITS SETS

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**ABSTRACT.** We consider some properties of the intersection of deleted digits Cantor sets with their translates. We investigate conditions on the set of digits such that, for any  $t$  between zero and the dimension of the deleted digits Cantor set itself, the set of translations such that the intersection has that Hausdorff dimension equal to  $t$  is dense in the set  $F$  of translations such that the intersection is non-empty. We make some simple observations regarding properties of the set  $F$ , in particular, we characterize when  $F$  is an interval, in terms of conditions on the digit set.

## 1. INTRODUCTION

Let  $n \geq 3$  be an integer. Any real number  $0 \leq x \leq 1$  can be written in base  $n$  as an  $n$ -ary expansion

$$(1.1) \quad x = 0.x_1x_2\cdots := \sum_{k=1}^{\infty} \frac{x_k}{n^k}$$

where  $x_k \in \{0, 1, \dots, n-1\}$ . This representation of a real number  $x$  in the interval  $[0, 1]$  is unique, except

$$0.x_1x_2\cdots x_k = 0.x_1x_2\cdots x_{k-1}y_ky_{k+1}\cdots$$

when  $x_k \neq 0$ ,  $y_k = x_k - 1$ , and  $y_j = n - 1$  for  $j > k$ .

Let  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  be a set of at least two integers, such that  $0 = d_1 < d_2 < \dots < d_m < n$  and  $m < n$ . The set

$$\mathcal{C} = \mathcal{C}_{n,\mathcal{D}} := \left\{ \sum_{k=1}^{\infty} \frac{x_k}{n^k} \mid x_k \in \mathcal{D} \right\}$$

is a *deleted digits Cantor set*. Consequently,  $\mathcal{C}_{n,\mathcal{D}}$  is obtained from the set of all  $n$ -ary representations (1.1) by restricting attention to those  $n$ -ary representations that only contain digits from the set  $\mathcal{D}$ , that is by deleting the digits not in  $\mathcal{D}$  from the set of all potential digits  $\{0, 1, \dots, n-1\}$ .

An  $n$ -ary *interval* is an interval of the form

$$\left[ \frac{k}{n^j}, \frac{k+1}{n^j} \right],$$

where  $j \geq 0$  is an integer and  $k = 0, 1, \dots, n^j - 1$ . Due to the structure of deleted digits Cantor sets, it is natural to work with the Minkowski dimension. Let  $S$  be a subset of the closed interval  $[0, 1]$ . Then the *Minkowski dimension* of  $S$  is

$$\dim_{\text{M}} S = \lim_{k \rightarrow \infty} \frac{\log \mathcal{N}_j(S)}{j \log n}$$

where  $\mathcal{N}_j(S)$  is the minimum number of  $n$ -ary intervals of length equal to  $1/n^j$  needed to cover  $S$ . If the limit does not exist, we can talk about the upper and lower Minkowski dimensions, obtained by replacing the limit by the limit superior and the limit inferior, respectively. Minkowski dimension is sometimes called *Minkowski–Bouligand dimension*, *box dimension*, *Kolmogorov dimension*, *entropy dimension*, or *limiting capacity*.

For a subset  $S$  of the interval  $[0, 1]$ , let

$$\Lambda_\beta(S) := \liminf_{\delta \rightarrow 0} \left\{ \sum_{I \in \mathcal{U}} |I|^\beta \right\}$$

where the infimum is over all coverings  $\mathcal{U}$  of  $S$  by  $n$ -ary intervals whose lengths  $|I|$  are at most  $\delta > 0$ . The *Hausdorff dimension* of  $S$  is the number  $\dim_{\text{H}} S$  satisfying

$$\begin{aligned} \Lambda_\beta(S) &= 0 \text{ for } \beta > \dim_{\text{H}} S, \text{ and} \\ \Lambda_\beta(S) &= \infty \text{ for } \beta < \dim_{\text{H}} S. \end{aligned}$$

The term Hausdorff dimension is sometimes replaced by *Hausdorff–Besicovitch dimension*, *Besicovitch dimension*, or *fractional dimension*. It is well-known, and not difficult to see, that this definition of the Hausdorff dimension agrees with the standard definition, where the infimum is over all countable covering of  $S$  by intervals of lengths at most  $\delta$ . For example, this was established by Besicovitch [1], when  $n = 2$ . See also the book [2] by Falconer. By the definition of Minkowski dimension, the Hausdorff dimension of  $S$  is bounded above by the (lower) Minkowski dimension of  $S$ . Hutchinson showed [3] both the Minkowski and Hausdorff dimensions of  $\mathcal{C}$  equals the similarity dimension  $\log_n m$ .

Let

$$\mathcal{F} := \{t \geq 0 \mid \mathcal{C} \cap (\mathcal{C} + t) \neq \emptyset\}.$$

be that set of positive real numbers  $t$ , such that the intersection of  $\mathcal{C}$  and its translate by  $t$ ,  $\mathcal{C} + t := \{x + t \mid x \in \mathcal{C}\}$ , is non-empty. Let  $\mathcal{F}_\alpha$  be the set of  $t$  in  $\mathcal{F}$  such that  $\mathcal{C} \cap (\mathcal{C} + t)$  has Hausdorff dimension  $\alpha \log_n m$ . Under suitable assumptions on  $\mathcal{D}$  we prove that  $\mathcal{F}_\alpha$  is dense in  $\mathcal{F}$  for any  $0 \leq \alpha \leq 1$ .

Our main results are

**Theorem 1.1.** *Let  $n$  be a positive integer and let  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  be a set of at least two integers, such that  $0 = d_1 < d_2 < \dots < d_m < n - 1$  and  $2 \leq d_{k+1} - d_k$ , for  $k = 1, 2, \dots, m - 1$ . Let  $\mathcal{C}$  be the real numbers of the form  $\sum_{k=1}^{\infty} x_k/n^k$  where each  $x_k$  is in  $\mathcal{D}$ . For each  $0 \leq \alpha \leq 1$ , the set  $\mathcal{M}_\alpha$  of all  $0 \leq t \leq 1$  such that the Hausdorff and Minkowski dimensions of  $\mathcal{C} \cap (\mathcal{C} + t)$  equals  $\alpha \log_n m$  is dense in the set  $\mathcal{F}$  of all  $0 \leq s \leq 1$  such that  $\mathcal{C} \cap (\mathcal{C} + s)$  is not the empty set.*

When  $\mathcal{C}$  is the triadic Cantor set, that is  $n = 3$  and  $\mathcal{D} = \{0, 2\}$ , the conclusions of Theorem 1.1 were established by Davis and Hu [4] and by a different method by Nekka and Li [5]. Of course, this does not follow from Theorem 1.1. To recover the conclusions of Theorem 1.1 for the triadic Cantor set we consider a class of “uniform” deleted digits Cantor sets.

A deleted digits Cantor set is *uniform* [6], [7], if there is an integer  $d \geq 2$ , such that  $d_j = d(j - 1)$ , for  $j = 1, 2, \dots, m$ . If  $d_m = n - 1$ , then we cannot apply Theorem 1.1. However, the additional structure on the digit set  $\mathcal{D}$ , imposed by assuming uniformity, allows us to establish the conclusions of Theorem 1.1 in this case. In fact, these conclusions hold under slightly weaker assumptions.

**Theorem 1.2.** *Let  $n$  be a positive integer and let  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  be a set of at least two integers, such that  $0 = d_1 < d_2 < \dots < d_m < n$ . Suppose there is an integer  $h > 1$ , such that each digit  $d_j$  is an integral multiple of  $h$ . Let  $\mathcal{C}$  be the real numbers of the form  $\sum_{k=1}^{\infty} x_k/n^k$  where each  $x_k$  is in  $\mathcal{D}$ . For each  $0 \leq \alpha \leq 1$ , the set  $\mathcal{M}_\alpha$  of all  $0 \leq t \leq 1$  such that the Hausdorff and Minkowski dimensions of  $\mathcal{C} \cap (\mathcal{C} + t)$  equals  $\alpha \log_n m$  is dense in the set  $\mathcal{F}$  of all  $0 \leq s \leq 1$  such that  $\mathcal{C} \cap (\mathcal{C} + s)$  is not the empty set.*

In Section 2 we show how the intersections  $\mathcal{C} \cap (\mathcal{C} + t)$  can be understood in term of a deleted intervals construction. This is used in Section 3 to show that if  $\mathcal{D}$  satisfies the separation condition  $d_{j+1} - d_j \geq 2$ , then for any  $0 \leq \alpha \leq 1$ , there is a  $t$  such that  $\mathcal{C} \cap (\mathcal{C} + t)$  has dimension  $\alpha \log_n m$ . The results in Sections 2 and 3 are then used in Sections 4 and 5 to establish Theorems 1.1 and 1.2 respectively. In Section 6 we investigate the geometry of  $\mathcal{F}$ , in particular, we obtain a characterization of when  $\mathcal{F}$  is an interval in terms of a property of the digit set  $\mathcal{D}$ . And as an application of our results, we construct everywhere discontinuous functions mapping the interval  $[0, 1]$  into itself. In Section 7 we investigate the necessity of the conditions imposed on the digit set  $\mathcal{D}$  in Theorem 1.1. In Section 8 we state some questions related to the results obtained in this paper.

For background information on fractal sets and their dimensions we refer the reader to the book [2] by Falconer. Motivations, including potential applications in physics, for studying the problems considered in this paper can, for example, be found in papers by Davis and Hu [4], Li and Nekka [7], and Dai and Tian [6].

## 2. PRELIMINARIES

The purpose of this section is to show how the deleted intervals construction of  $\mathcal{C}$  can be used to analyse  $\mathcal{C} \cap (\mathcal{C} + t)$ . These observations form the basis for our proof of Theorem 1.1.

**2.1. Constructions of deleted digits Cantor sets.** Let

$$\mathcal{D} = \{d_1, d_2, \dots, d_m\}$$

be a set of at least two distinct integers, the set of *digits*, such that  $0 = d_1 < d_2 < \dots < d_m < n$ . The corresponding *deleted digits Cantor set*  $\mathcal{C}$  is the set of  $n$ -ary real numbers in  $[0, 1]$  that can be constructed using only digits from the digit set  $\mathcal{D}$ , that is

$$\mathcal{C} = \left\{ 0.x_1x_2 \dots \mid x_j \in \mathcal{D} \right\}.$$

**2.1.1. Self-similarity construction of  $\mathcal{C}$ .** Let  $S_j(x) := (x + d_j)/n$  for  $j = 1, \dots, m$ . Let  $\mathcal{C}_0 := [0, 1]$ , and inductively

$$\mathcal{C}_{k+1} := \bigcup_{j=1}^m S_j(\mathcal{C}_k).$$

Then  $\mathcal{C}_k = \{0.x_1x_2 \dots \mid x_j \in \mathcal{D} \text{ for } j \leq k\}$  is the set of real numbers in the interval  $[0, 1]$  that admit an  $n$ -ary representation whose first  $k$  digits are chosen from the digit set  $\mathcal{D}$ . Consequently,  $\mathcal{C} = \bigcap_{k=0}^{\infty} \mathcal{C}_k$ . Furthermore, for each  $k$ , the set  $\mathcal{C}_k$  consists of  $m^k$  closed intervals each of length  $1/n^k$  and these intervals are  $n$ -ary intervals with disjoint interiors.

2.1.2. *Retained/deleted intervals construction of  $\mathcal{C}$ .* If  $I = [a, b]$  is a closed interval, we can consider the partition,

$$I_j := \left[ a + \frac{j-1}{n}(b-a), a + \frac{j}{n}(b-a) \right],$$

$j = 1, 2, \dots, n$ , of  $I$  into  $n$  closed subintervals of equal length. The subset of  $I$ , obtained by retaining the intervals in the partition corresponding to digits in  $\mathcal{D}$ , that is the set

$$\bigcup_{j=1}^m \left[ a + \frac{d_j}{n}(b-a), a + \frac{d_j+1}{n}(b-a) \right]$$

is a *refinement* of the interval  $[a, b]$ . With this terminology  $\mathcal{C}_{k+1}$  is obtained from  $\mathcal{C}_k$  by refining each interval in  $\mathcal{C}_k$ .

2.2. **Investigating  $\mathcal{C} \cap (\mathcal{C} + x)$ .** In the remainder of this section we will assume  $\mathcal{D}$  satisfies the *separation condition*  $2 \leq d_{k+1} - d_k$  for  $k = 1, 2, \dots, m-1$ . Note that

$$\mathcal{C} \cap (\mathcal{C} + x) = \bigcap_{k=0}^{\infty} (\mathcal{C}_k \cap (\mathcal{C}_k + x)).$$

For  $x = 0.x_1x_2\cdots$  let  $\lfloor x \rfloor_k$  denote the truncation to the first  $k$  places, that is,

$$\lfloor x \rfloor_k := 0.x_1x_2\cdots x_k.$$

If  $x$  admit a finite  $n$ -ary representation  $\lfloor x \rfloor_k$  depends on which  $n$ -ary representation is chosen. We will consider  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  in place of  $\mathcal{C}_k \cap (\mathcal{C}_k + x)$ , since both  $\mathcal{C}_k$  and  $\mathcal{C}_k + \lfloor x \rfloor_k$  consists of  $n$ -ary intervals of lengths  $1/3^k$ .

Below, *an interval in  $\mathcal{C}_j$* , is short for an  $n$ -ary interval in  $\mathcal{C}_j$  of length  $1/n^j$ , that is one of the interval obtained by applying the refinement process. A similar convention applies to the term *an interval in  $\mathcal{C}_j + y$* .

We will investigate how  $\mathcal{C}_{k+1} \cap (\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1})$  is related to  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  for  $k \geq 0$ . Recall that  $\mathcal{C}_{k+1}$  is obtained from  $\mathcal{C}_k$  by refining each interval in  $\mathcal{C}_k$ . Consequently,  $\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1}$  is obtained from  $\mathcal{C}_k + \lfloor x \rfloor_k$  by refining each interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  and then translating the resulting intervals to the right by  $x_{k+1}/n^{k+1}$ .

Let  $I$  be one of the intervals in  $\mathcal{C}_k$ . We will consider what happens to  $I$  as we *transition* from  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  to  $\mathcal{C}_{k+1} \cap (\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1})$ . Since  $\lfloor x \rfloor_k$  is an integral multiple of  $1/n^k$ , the intervals in  $\mathcal{C}_k + \lfloor x \rfloor_k$  either coincides with intervals in  $\mathcal{C}_k$ , they have one or both endpoints in common with intervals in  $\mathcal{C}_k$ , or are at least  $1/n^k$  units away from any interval in  $\mathcal{C}_k$ . Hence there are four possibilities for the interval  $I$  to consider.

- $I$  is in the *interval case*, if there an interval  $J$  in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that  $I = J$ .
- $I$  is in the *potential interval case*, if there is an interval  $J$  in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that the left-hand endpoint of  $J$  is the right-hand endpoint of  $I$ .
- $I$  is in the *potentially empty case*, if there is an interval  $J$  in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that the left-hand endpoint of  $J$  is the right-hand endpoint of  $I$ .
- $I$  is in the *empty case*, if  $I$  does not intersect any interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$ .

It is possible for  $I$  to be simultaneously be in the potential interval case and the potentially empty case. Due to the separation condition it is not possible for an interval in  $\mathcal{C}_k$  to simultaneously be in the interval case and the potential interval case, or simultaneously in the interval case and the potentially empty case.

For any  $x$

$$(2.1) \quad \mathcal{C} \cap (\mathcal{C} + x) = \bigcap_{k=0}^{\infty} \bigcup I,$$

where the union is over the intervals in  $\mathcal{C}_k$  that are not in the empty case.

We will describe the four cases above in more detail, under the assumption that  $x = 0.x_1x_2\cdots$  does not terminate in repeating 0's or in repeating  $n-1$ 's, equivalently,  $0 < x - \lfloor x \rfloor_k < 1/n^k$  for all  $k$ , in particular, we will see that we can exclude the intervals in  $\mathcal{C}_k$  that are in the potentially empty case from the union in (2.1).

2.2.1. *Suppose  $I$  is in the interval case.* Let  $J$  be the interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that  $I = J$ . Then  $I \cap (J + x - \lfloor x \rfloor_k)$  is an interval of length  $\frac{1}{n^k} - (x - \lfloor x \rfloor_k)$  hence the refinement/translation process applied to the intervals  $I$  and  $J$  may lead to points in  $\mathcal{C} \cap (\mathcal{C} + x)$ .

2.2.2. *Suppose  $I$  is in the potential interval case.* Let  $J$  be the interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that the left-hand endpoint of  $J$  is the right-hand endpoint of  $I$ . Then  $I \cap (J + x - \lfloor x \rfloor_k)$  is an interval of length  $x - \lfloor x \rfloor_k$  hence the refinement/translation process applied to the intervals  $I$  and  $J$  may lead to points in  $\mathcal{C} \cap (\mathcal{C} + x)$ .

2.2.3. *Suppose  $I$  is in the potentially empty case.* Let  $J$  be the interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that the right-hand endpoint of  $J$  is the left-hand endpoint of  $I$ . Since  $0 < x - \lfloor x \rfloor_k$  we have  $I \cap (J + x - \lfloor x \rfloor_k) = \emptyset$ , so this intersection does not lead to points in  $\mathcal{C} \cap (\mathcal{C} + x)$ .

2.2.4. *Suppose  $I$  is in the empty case.* Then  $I \cap (\mathcal{C}_k + \lfloor x \rfloor_k) = \emptyset$ . Since the interval  $I$  is at least  $1/n^k$  units away from any interval  $\mathcal{C}_k + \lfloor x \rfloor_k$  in and  $x - \lfloor x \rfloor_k < 1/n^k$  we have  $I \cap (\mathcal{C}_k + x) = \emptyset$ . So  $I$  does not contribute points to  $\mathcal{C} \cap (\mathcal{C} + x)$ .

*Remark 2.1.* If  $I$  is any interval in  $\mathcal{C}_k$  and  $J$  is any interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  disjoint from  $I$ , then  $I$  and  $J + x - \lfloor x \rfloor_k$  are also disjoint. The reason for this is that the separation condition  $d_{j+1} - d_j \geq 2$  implies the distance between intervals  $I$  and  $J$  is an integral multiple of  $1/n^k$  and  $0 \leq x - \lfloor x \rfloor_k < 1/n^k$ . Hence, when we consider  $\mathcal{C} \cap (\mathcal{C} + x)$ , we can ignore intervals  $J$  that do not occur in the first two cases above. (The first three cases, if we allow  $x$  to admit a finite  $n$ -ary representation.)

2.3. **A Description of  $\mathcal{F}$ .** The following description of  $\mathcal{F}$  is useful below. Let  $x$  be an element of  $\mathcal{F}$  such that  $0 < x < 1$ .

2.3.1. *Suppose  $x$  does not have a finite  $n$ -ary expansion.* Then  $0 < x - \lfloor x \rfloor_k < 1/n^k$  for all  $k$ , hence we can apply the analysis at the end of sub-section 2.2. In particular, for all  $k$ , at least one of the intervals in  $\mathcal{C}_k$  will either be in the interval case or in the potential interval case.

2.3.2. *Suppose  $x$  has a finite  $n$ -ary expansion.* Then we can write  $x = 0.x_1x_2\cdots x_k$  where  $x_k \neq 0$ . Then

$$\mathcal{C} \cap (\mathcal{C} + x) = \bigcap_{j=k}^{\infty} (\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_k)),$$

since  $x = \lfloor x \rfloor_k$ . Consequently, if one of the intervals in  $\mathcal{C}_k$  is in the interval case, then repeated refinement of that interval leads to a subset of  $\mathcal{C} \cap (\mathcal{C} + x)$  that is similar to  $\mathcal{C}$ .

On the other hand, if one of the intervals  $I$  in  $\mathcal{C}_k$  is in the potential interval case or in the potentially empty case, then there is an interval  $J$  in  $\mathcal{C}_k + x$  such that  $I \cap J$  contains exactly one point,  $y$  say. If  $d_m = n - 1$ , then  $y$  will be contained in the intersection of the refinements of  $I$  and  $J$ . Hence  $y$  will be a point in  $\mathcal{C} \cap (\mathcal{C} + x)$  and any other point in  $\mathcal{C} \cap (\mathcal{C} + x)$  will be at least  $1/n^k$  units away from  $y$ . On the other hand, if  $d_m < n - 1$ , then the refinements of  $I$  and  $J$  will not intersect. In particular,  $y$  will not be a point in  $\mathcal{C} \cap (\mathcal{C} + x)$ .

*Remark 2.2.* The description above shows that, if  $x$  has a finite  $n$ -ary expansion, then

$$\mathcal{C} \cap (\mathcal{C} + x) = E \cup F,$$

where  $E$  is a finite, perhaps empty, union of sets similar to  $\mathcal{C}$  and  $F$  is a finite, perhaps empty, set. Hence, if  $x$  has a finite  $n$ -ary expansion, then the dimension  $\mathcal{C} \cap (\mathcal{C} + x)$  is either 0 or  $\log m / \log n$ . Consequently, to prove Theorem 1.1 we must consider  $x$  that do not have finite  $n$ -ary expansions.

### 3. A STEPPING STONE

The following provides the key step in the proofs of Theorem 1.1 and Theorem 1.2 and is, perhaps, of independent interest.

**Proposition 3.1.** *If  $\mathcal{D}$  satisfies the separation condition  $d_{j+1} - d_j \geq 2$  for all  $j = 1, 2, \dots, m - 1$ , then given any  $0 \leq \alpha \leq 1$ , there is an  $x$  in  $\mathcal{F}$  such that  $\mathcal{C} \cap (\mathcal{C} + x)$  has Minkowski and Hausdorff dimension equal to  $\alpha \log_n m$ . This  $x$  may be chosen not to admit a terminating  $n$ -ary representation.*

Let  $0 \leq \alpha \leq 1$  be given. The proof is completed in two steps. First we use the transition process to construct an  $x$  such that  $\mathcal{C} \cap (\mathcal{C} + x)$  has Minkowski dimension  $\alpha \log_n m$ . Then we show that for this  $x$  the set  $\mathcal{C} \cap (\mathcal{C} + x)$  also has Hausdorff dimension  $\alpha \log_n m$ .

**3.1. Construction of  $x$  using the Minkowski dimension as a guide.** We begin the refinement process in the interval case  $\mathcal{C}_0 \cap (\mathcal{C}_0 + 0)$ . The idea of the proof is, if  $x_{j+1} = 0$ , then transitioning from  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  to  $\mathcal{C}_{j+1} \cap (\mathcal{C}_{j+1} + \lfloor x \rfloor_{j+1})$  multiplies the number of interval cases by  $m$ , and if  $x_{j+1} = d_m$  the transition multiplies the number of interval cases by one. In either case no potential interval cases or potentially empty cases appear.

Let  $h_j := \lfloor j\alpha \rfloor$ , then  $h_j$  is a positive integer such that  $h_j \leq j\alpha < 1 + h_j$ , and consequently,  $h_j/j \rightarrow \alpha$  as  $j \rightarrow \infty$ . Since  $0 \leq \alpha \leq 1$  we have  $h_j \leq h_{j+1} \leq 1 + h_j$ . Suppose  $0 < \alpha < 1$ . For  $j \geq 1$  set

$$x_j = \begin{cases} d_m & \text{if } h_j = h_{j-1} \\ 0 & \text{if } h_j = 1 + h_{j-1} \end{cases}.$$

Then the number of interval cases in  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  is  $m^{h_j}$ . Since  $\mathcal{C} \cap (\mathcal{C} + x)$  is a subset of  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  this provides an upper bound for the number of interval of length  $1/n^j$  needed to cover  $\mathcal{C} \cap (\mathcal{C} + x)$ :

$$(3.1) \quad \mathcal{N}_j(\mathcal{C} \cap (\mathcal{C} + x)) \leq m^{h_j}.$$

To calculate the Minkowski dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  it remains to check that any interval case in  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  leads to points in  $\mathcal{C} \cap (\mathcal{C} + x)$  so that the upper bound (3.1) for  $\mathcal{N}_j(\mathcal{C} \cap (\mathcal{C} + x))$  is also a lower bound. But by the refinement process each interval in  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  transitions to one or  $m$  sub-intervals in  $\mathcal{C}_{j+1} \cap (\mathcal{C}_{j+1} + \lfloor x \rfloor_{j+1})$ . Hence, it follows from the Nested Interval Theorem that each interval in  $\mathcal{C}_j \cap (\mathcal{C}_j + \lfloor x \rfloor_j)$  has infinitely many points in common with  $\mathcal{C} \cap (\mathcal{C} + x)$ . Using  $h_j/j \rightarrow \alpha$ , we conclude

$$(3.2) \quad \frac{\log \mathcal{N}_j(\mathcal{C} \cap (\mathcal{C} + x))}{j \log n} = \frac{h_j \log m}{j \log n} \rightarrow \alpha \frac{\log m}{\log n}$$

as  $j \rightarrow \infty$ .

We can change some of the digits  $x_j = 0$  to  $x_j = d_m$  or visa versa, as long as the limit (3.2) remains unchanged. That is, we can make changes of this nature on a sparse set of  $j$ 's. In particular, if necessary, we can ensure that  $x$  neither terminates in repeating 0's nor in repeating  $d_m$ 's. In particular, this observation allows us to deal with the cases  $\alpha = 0$  and  $\alpha = 1$ , using arguments presented above. The details are left for the reader.

**3.2. Hausdorff dimension.** It is not immediate that the Hausdorff dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  equals its Minkowski dimension because the set  $\mathcal{C} \cap (\mathcal{C} + x)$  need not be self-similar, see Section 8. Consequently, it remains to check that the Hausdorff dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is bounded below by  $\alpha \log_n m$ . The argument below is inspired by an argument due to Eggleston [8].

Let  $\mathcal{A}$  be the collection of  $n$ -ary intervals introduced as part of the construction of  $x$ . Then, the subcollection  $\mathcal{A}_j$  of  $n$ -ary intervals in  $\mathcal{A}$  of length  $1/n^j$  contains  $m^{h_j}$  members, where  $h_0 = 0$ , and  $h_j/j \rightarrow \alpha$  as  $j \rightarrow \infty$ . And, by equality in (3.1), each interval in  $\mathcal{A}_j$  refines to  $m^{h_{j+1}-h_j}$  intervals in  $\mathcal{A}_{j+1}$ . Let  $\beta < \alpha \log_n m$ . Then,  $m^{h_j/j} \rightarrow m^\alpha$  implies  $\sum_{j=0}^{\infty} n^{j\beta}/m^{h_j} < \infty$ . Let  $N$  be an integer such that

$$(3.3) \quad \sum_{j=N}^{\infty} \frac{n^{j\beta}}{m^{h_j}} < \frac{1}{2}.$$

Let  $\mathcal{U}$  be a collection of  $n$ -ary intervals covering  $\mathcal{C} \cap (\mathcal{C} + x)$  and whose lengths are at most  $1/n^N$ . We will show that  $\sum_{I \in \mathcal{U}} |I|^\beta \geq 1$ . Consequently, the Hausdorff dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is bounded below by  $\beta$ . Since  $\beta < \alpha \log_n m$  is arbitrary, it follows that the Hausdorff dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is bounded below by  $\alpha \log_n m$ . The proof is by contradiction. Suppose

$$(3.4) \quad \sum_{I \in \mathcal{U}} |I|^\beta < 1.$$

Remove the intervals from  $\mathcal{U}$  that do not intersect  $\mathcal{C} \cap (\mathcal{C} + x)$ . If an interval  $I$  in  $\mathcal{U}$  is not in  $\mathcal{A}$ , but shares an endpoint with an interval  $J$  in  $\mathcal{A}$ , replace  $I$  by  $J$ . Making these changes to  $\mathcal{U}$  will not increase the sum in (3.4), hence we can assume  $\mathcal{U}$  is a subset of  $\mathcal{A}$ . Let  $\mathcal{U}_j$  be the intervals in  $\mathcal{U}$  of length  $1/n^j$ . By convergence of the sum in (3.4) the set  $\mathcal{U}_j$  is finite. If  $\#\mathcal{U}_j$  denotes the number of intervals in  $\mathcal{U}_j$ , then

$$(\#\mathcal{U}_j) \frac{1}{n^{j\beta}} = \sum_{I \in \mathcal{U}_j} |I|^\beta < 1$$

by (3.4). Consequently,

$$(3.5) \quad \sum_{I \in \mathcal{U}_j} |I| = (\#\mathcal{U}_j) \frac{1}{n^j} < \frac{n^{j\beta}}{n^j}.$$

If  $j \leq k$ , then any interval in  $\mathcal{U}_j$  is an interval in  $\mathcal{A}_j$  and refines to  $m^{h_k-h_j}$  intervals in  $\mathcal{A}_k$ . Hence every interval in  $\mathcal{U}_j$  covers exactly  $m^{h_k-h_j}$  intervals in  $\mathcal{A}_k$ . So, if  $\mathcal{B}_{j,k}$  is the intervals in  $\mathcal{A}_k$  covered by intervals in  $\mathcal{U}_j$ , then

$$\sum_{I \in \mathcal{B}_{j,k}} |I| = \frac{m^{h_k-h_j}}{n^{k-j}} \sum_{J \in \mathcal{U}_j} |J|.$$

The factor  $1/n^{k-j}$  appears since  $|I| = 1/n^k$  and  $|J| = 1/n^j$ . So by (3.5)

$$\sum_{I \in \mathcal{B}_{j,k}} |I| < \frac{m^{h_k-h_j}}{n^{k-j}} \frac{n^{j\beta}}{n^j}.$$

Hence

$$\sum_{j=N}^k \sum_{I \in \mathcal{B}_{j,k}} |I| < \frac{m^{h_k}}{n^k} \sum_{j=N}^k \frac{n^{j\beta}}{m^{h_j}} < \frac{1}{2} \frac{m^{h_k}}{n^k} = \frac{1}{2} \sum_{I \in \mathcal{A}_k} |I|.$$

Where the last inequality used (3.3). Consequently, there are intervals in  $\mathcal{A}_k$  disjoint from  $\cup_{j=N}^k \cup_{I \in \mathcal{U}_j} I$ . Let  $\mathcal{H}_k$  be the union of the intervals in  $\mathcal{A}_k$  that are disjoint from  $\cup_{j=N}^k \cup_{I \in \mathcal{U}_j} I$ . The  $\mathcal{H}_{k+1}$  is a subset of  $\mathcal{H}_k$ , hence, by compactness,  $\cap_{k=N}^\infty \mathcal{H}_k$  is non-empty. Any point in  $\cap_{k=N}^\infty \mathcal{H}_k$  is a point in  $\mathcal{C} \cap (\mathcal{C} + x)$  not covered by any interval in  $\mathcal{U}$ . Contradicting that  $\mathcal{U}$  is a cover of  $\mathcal{C} \cap (\mathcal{C} + x)$ .

We have shown  $\dim_M \mathcal{C} \cap (\mathcal{C} + x) \leq \dim_H \mathcal{C} \cap (\mathcal{C} + x)$ . Consequently, the Hausdorff and Minkowski dimensions of  $\mathcal{C} \cap (\mathcal{C} + x)$  are equal.

#### 4. PROOF OF THEOREM 1.1

Since  $d_m < n - 1$ , the largest element of  $\mathcal{C}$ , that is  $0.d_m d_m d_m \dots$ , is  $< 1$ . Hence, if  $0 \leq y \leq 1$  is such that  $\mathcal{C} \cap (\mathcal{C} + y)$  is non-empty, then  $y < 1$ .

Let  $0 \leq y < 1$  be such that  $\mathcal{C} \cap (\mathcal{C} + y)$  is non-empty. Let  $\varepsilon > 0$  be given. Write  $y = 0.y_1 y_2 \dots$ . Pick  $k$  so large that  $1/n^k < \varepsilon$ . Let  $x_j = y_j$  for  $j = 1, 2, \dots, k$ . Then no matter how we determine  $x_j$  for  $j > k$ , we have  $|x - y| < \varepsilon$ , where  $x = 0.x_1 x_2 \dots$ . We will ensure that  $x$  does not have a terminating  $n$ -ary expansion.

**4.1. Interval Cases.** Suppose there is an interval in  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$ , then at least one interval in  $\mathcal{C}_k$  is in the interval case. Setting  $x_{k+1} = 0$ , each interval in  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  transitions to  $m$  intervals in  $\mathcal{C}_{k+1} \cap (\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1})$  and, using that  $d_m < n - 1$  we see that all potential interval cases and potentially empty cases transitions to empty cases. Hence we now have interval cases only. The transition to interval cases only, is illustrated in Figure 4.1, in the case where  $n = 7$  and  $\mathcal{D} = \{0, 3, 5\}$ . The two long lines represents the interval  $I$  in  $\mathcal{C}_k$  and the interval  $J$  in  $\mathcal{C}_k + \lfloor x \rfloor_k$ . Depending on whether the line on the left is  $I$  or  $J$  Figure 4.1 illustrates the potentially empty or the potential interval case. The heavier lines are the refinements of the two intervals. For clarity one line is slightly elevated compared to the other line.

We are left with a finite number of interval cases in  $\mathcal{C}_{k+1}$ . We can apply Proposition 3.1 to, a self-similar copy of, one of these intervals cases to arrive at the  $x_j$  for  $j > k + 1$ . Since all the interval cases essentially are identical, it follows that



FIGURE 4.1. Setting  $x_{k+1} = 0$  and refining the intervals

$\mathcal{C} \cap (\mathcal{C} + x)$  is a finite union of sets whose Minkowski and Hausdorff dimensions both equal  $\alpha \log_n m$ .

**4.2. Remaining Cases.** Suppose  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  does not contain an interval, but does contain potential interval cases. Let  $I$  be an interval in  $\mathcal{C}_k$  and  $J$  an interval in  $\mathcal{C}_k + \lfloor x \rfloor_k$  such that the right endpoint of  $J$  coincides with the left endpoint of  $I$ . Setting  $x_{k+1} = n - d_m$ , then translates the right-most interval in the refinement of  $J$  onto the left-most interval in the refinement of  $I$ . Hence  $\mathcal{C}_{k+1} \cap (\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1})$  contains an interval and we can now repeat the previous argument.

Finally, suppose  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  does not contain an interval nor a potential interval case, but does contain a potentially empty case. Then  $\mathcal{C}_{k+1} \cap (\mathcal{C}_{k+1} + \lfloor x \rfloor_{k+1})$  will be empty even if  $x_{k+1} = 0$ , because  $d_m < n - 1$ . Contradicting that  $\mathcal{C} \cap (\mathcal{C} + y)$  is non-empty.

## 5. PROOF OF THEOREM 1.2

If  $d_m < n - 1$ , this is a special case of Theorem 1.1, hence we may assume that  $d_m = n - 1$ .

The proof is similar to the proof of Theorem 1.1, we will comment on the differences. If there is an interval in  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$ , then there are no potential interval cases or potentially empty cases. Hence there is no reason to begin by setting  $x_{k+1} = 0$ . The rest of the proof in the interval case is unchanged.

Suppose  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  does not contain an interval, but does contain potential interval cases. In this case the argument remains unchanged. Note that  $x_{k+1} = 1$ .

Finally, suppose  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor x \rfloor_k)$  does not contain an interval nor a potential interval case, but does contain a potentially empty case. In this case the argument presented above does not work. Note  $y_j = 0$  for all  $j > k$ , because otherwise  $\mathcal{C} \cap (\mathcal{C} + y)$  would be empty. Also,  $y > 0$ , since we are not in the interval case. Let  $l$  be the largest subscript such that  $y_l \neq 0$ . Let  $z = 0.z_1z_2\cdots$ , where  $z_j = y_j$  when  $j < l$ ,  $z_l = y_l - 1$ , and  $z_j = n - 1$  when  $l < j$ . Then  $\lfloor z \rfloor_k \rightarrow y$  and  $\mathcal{C}_k \cap (\mathcal{C}_k + \lfloor z \rfloor_k)$  is in the interval case for all  $k \geq l$ . Hence, we can, once again, apply the interval case argument.

## 6. GEOMETRY OF $\mathcal{F}$ .

Let  $m < n$  be positive integers and let  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$  be a set of at least two integers, such that  $0 = d_1 < d_2 < \cdots < d_m < n$  and let  $\mathcal{C}$  be the corresponding deleted digits Cantor set. The set  $\mathcal{F}$  of all  $x$  in the interval  $[0, 1]$ , such that  $\mathcal{C} \cap (\mathcal{C} + x)$  is non-empty plays a starring role in Theorem 1.1. The main purpose of this section is to investigate the geometry of  $\mathcal{F}$ .

Since  $\mathcal{C}$  is compact,  $\mathcal{F}$  is also compact:

**Lemma 6.1.**  *$\mathcal{F}$  is compact and non-empty.*

*Proof.* Clearly 0 is in  $\mathcal{F}$  and  $\mathcal{F}$  is bounded, since  $\mathcal{C}$  is bounded. Hence it is sufficient to show that  $\mathcal{F}$  is closed. Let  $x > 0$ . Suppose  $\mathcal{C} \cap (\mathcal{C} + x)$  is empty. By compactness of  $\mathcal{C}$ ,

$$\varepsilon := \text{dist}(\mathcal{C}, \mathcal{C} + x) > 0.$$

Hence,  $\mathcal{C} \cap (\mathcal{C} + y)$  is empty when  $|x - y| < \varepsilon$ . Consequently, the complement of  $\mathcal{F}$  in the interval  $[0, \infty)$  is an open set.  $\square$

We begin by showing that  $\mathcal{G} := (-\mathcal{F}) \cup \mathcal{F}$  is the attractor for a set of similarity transformations.

Let  $\Delta := \mathcal{D} - \mathcal{D} = \{d - e \mid d, e \in \mathcal{D}\}$  and

$$(6.1) \quad \sigma_\delta(x) := \frac{x + \delta}{n}$$

for  $\delta \in \Delta$ . Since 0 is in  $\mathcal{D}$ , we have  $\pm\mathcal{D} \subseteq \Delta$ . Since  $\sigma_{d_j} = S_j$  for  $j = 1, 2, \dots, m$ . This family of contractive similarities is closely related to the similarities used above to generate  $\mathcal{C}$ .

**Lemma 6.2.**  *$\mathcal{G}$  is the unique non-empty compact set invariant under the contractions  $\sigma_\delta$ ,  $\delta \in \Delta$ .*

*Proof.* Note  $\mathcal{G}$  is the set of real numbers  $x$  such that  $\mathcal{C} \cap (\mathcal{C} + x)$  is non-empty. Consequently,  $\mathcal{G} = \mathcal{C} - \mathcal{C} = \{s - t \mid s, t \in \mathcal{C}\}$ . Then the self-similarity construction of  $\mathcal{C}$  leads to

$$\begin{aligned} \mathcal{C} - \mathcal{C} &= \bigcup_{d, e \in \mathcal{D}} \sigma_d(\mathcal{C}) - \sigma_e(\mathcal{C}) \\ &= \bigcup_{\delta \in \Delta} \sigma_\delta(\mathcal{C} - \mathcal{C}) \end{aligned}$$

where the second equality used  $\sigma_d(x) - \sigma_e(y) = \sigma_{d-e}(x - y)$ .  $\square$

This leads to a characterization of when  $\mathcal{F}$  is an interval:

**Proposition 6.3.** *Write  $\Delta = \{\delta_j \mid j = 1, 2, \dots, M\}$ , where  $\delta_j < \delta_{j+1}$  for  $j = 1, 2, \dots, M - 1$ . Then  $\mathcal{F}$  is an interval iff*

$$2d_m \geq (n - 1)(\delta_{j+1} - \delta_j)$$

*for all  $j = 1, 2, \dots, M - 1$ . If  $\mathcal{F}$  is an interval, then  $\mathcal{F} = [0, d_m/(n - 1)]$ .*

*Proof.* The largest element of  $\mathcal{C}$  is  $d_m/(n - 1)$ , hence the smallest interval containing  $\mathcal{G}$  is

$$I := [-d_m/(n - 1), d_m/(n - 1)].$$

So  $\mathcal{G}$  is an interval iff  $\mathcal{G} = I$ . By construction of  $I$  we have  $\bigcup_{\delta \in \Delta} \sigma_\delta(I) \subseteq I$ . Since  $\sigma_\delta(\mathcal{G}) \subseteq \sigma_\delta(I)$ , each of the intervals  $\sigma_\delta(I)$ ,  $\delta \in \Delta$  has points in common with  $\mathcal{G}$ . Consequently,  $\mathcal{G}$  is an interval iff the intervals  $\sigma_{\delta_j}(I)$  and  $\sigma_{\delta_{j+1}}(I)$  overlap for all  $j = 1, 2, \dots, M - 1$ . Hence  $\mathcal{G}$  is an interval iff

$$\sigma_{\delta_j}(d_m/(n - 1)) \geq \sigma_{\delta_{j+1}}(-d_m/(n - 1))$$

for all  $j = 1, 2, \dots, M - 1$ . By (6.1) this is equivalent to the condition listed above.  $\square$

**Corollary 6.4.** *If  $\mathcal{D}$  consists of even integers, then  $\mathcal{F}$  is an interval if and only if*

$$\mathcal{D} - \mathcal{D} = \{-n + 1, \dots, -2, 0, 2, \dots, n - 1\}.$$

*In the affirmative case  $n$  is odd and  $\mathcal{F} = [0, 1]$ .*

**Example 6.5.** It is a simple consequence of Corollary 6.4 that  $\mathcal{F} = [0, 1]$ , if  $n = 7$  and  $\mathcal{D} = \{0, 2, 6\}$ .

**Example 6.6.** Suppose  $d_j = 2(j-1)$  for  $j = 1, 2, \dots, m$  and  $2(m-1) = n-1$ , then  $\mathcal{F} = [0, 1]$  by Corollary 6.4. Let  $f(t) := \underline{\dim}_M(\mathcal{C} \cap (\mathcal{C} + t))$  be the lower Minkowski dimension of  $\mathcal{C} \cap (\mathcal{C} + t)$ . By Theorem 1.2  $f$  maps any subinterval of  $[0, 1]$  onto the interval  $[0, \log m / \log n]$ . In particular,  $f$  is discontinuous at every point in the interval  $[0, 1]$ .

If there is a bounded open interval  $I$  such that

$$(6.2) \quad \bigcup_{\delta \in \Delta} \sigma_\delta(I) \subseteq I \quad (\text{disjoint union})$$

then  $\mathcal{G}$  is a subset of the closure of  $I$ . Hence, it follows from the proof of Proposition 6.3, that there is a bounded *open* interval  $I$ , such that (6.2) if and only if

$$(6.3) \quad 2d_m \leq (n-1)(\delta_{j+1} - \delta_j)$$

for all  $j = 1, 2, \dots, M-1$ . In the affirmative case

$$I = (-d_m/(n-1), d_m/(n-1)).$$

In particular, (6.3) implies that  $\sigma_\delta$ ,  $\delta \in \Delta$  satisfies the open set condition. Now  $\mathcal{D}$  has  $m$  elements and  $\pm\mathcal{D} \subseteq \Delta$ , so, since each digit in  $\mathcal{D}$  is non-negative,  $\Delta$  has at least  $2m-1$  elements. Hence the sparcity condition (6.3) implies

$$\dim_H \mathcal{F} = \log_n \# \Delta > \log_n \# \mathcal{D} = \dim_H \mathcal{C}$$

by Hutchinson [3], [2].

The following explores the structure of  $\mathcal{F}$  under the conditions of Theorem 1.2. These conditions imply (6.3).

**Proposition 6.7.** *If there is an integer  $h > 1$  dividing all the digits in  $\mathcal{D}$ , then  $\mathcal{F}$  is the interval  $[0, 1]$  or there is a deleted digits Cantor set  $\mathcal{B}$  such that  $(-\mathcal{F}) \cup \mathcal{F} = h\mathcal{B} - d_m/(n-1)$ .*

*Proof.* Suppose  $\mathcal{D} = \{d_1, d_2, \dots, d_m\}$ , where  $e_j = d_j/h$  is an integer for  $j = 1, 2, \dots, m$ . Fix  $t \in \mathcal{G}$ . Suppose  $x, y$  in  $\mathcal{C}$  are such that  $x = y + t$ . That is such that  $t = x - y$ . Consider the  $n$ -ary representations  $x = 0.x_1x_2\dots$  and  $y = 0.y_1y_2\dots$ . Rewrite  $(x - y)/h = t/h$  as

$$\sum_{k=1}^{\infty} \frac{(x_k - y_k)/h}{n^k} = \frac{t}{h}.$$

By making suitable choices for  $x$  and  $y$  we can arrange that  $(x_k - y_k)/h$  is any sequence of points in  $\mathcal{E} = \{e_i - e_j \mid 1 \leq i, j \leq m\}$ . So setting  $z_k = e_m + (x_k - y_k)/h$  we can arrange that  $z_k$  is any sequence in  $\mathcal{E} + e_m$ . Consequently,  $z = 0.z_1z_2\dots$  can be any number in the set  $\mathcal{B}$  of  $n$ -ary numbers with digits in  $\mathcal{E} + e_m$ . Thus  $t/h$  can be any number number in  $\mathcal{B} - \frac{e_m}{n-1}$ . Hence

$$\mathcal{G} = h\mathcal{B} - \frac{d_m}{n-1}.$$

Note,  $\{0, e_m, 2e_m\} \subseteq \mathcal{E} + e_m \subseteq \{0, 1, \dots, 2e_m\}$ .

If  $\mathcal{E} + e_m = \{0, 1, \dots, n-1\}$ , then  $\mathcal{B}$  is the interval  $[0, 1]$ . Otherwise,  $\mathcal{B}$  is the deleted digits Cantor set consisting of the  $n$ -ary numbers with digits from  $\mathcal{E} + e_m$ .  $\square$

## 7. EXAMPLES

In this section we will present some examples illustrating that to obtain the conclusions of Theorem 1.1 we must impose some conditions on the digit set  $\mathcal{D}$ . In Theorem 1.1 we impose two condition on  $\mathcal{D}$ . The separation condition  $d_{j+1} - d_j \geq 2$  and the condition  $d_m < n - 1$ . In Theorem 1.2 the condition  $d_m < n - 1$  is replaced by a uniformity condition.

In the first example  $\mathcal{D}$  violates both of the conditions imposed in Theorem 1.1.

**Example 7.1.** For any  $n \geq 2$ , if  $\mathcal{D} = \{0, 1, \dots, n - 1\}$ , then  $\mathcal{C}$  is the closed interval  $[0, 1]$ . Consequently,  $\mathcal{C} \cap (\mathcal{C} + x)$  either has dimension zero or one.

In the following example,  $\mathcal{D}$  does not satisfy the separation condition  $d_{j+1} - d_j \geq 2$ , but  $d_m < n - 1$ .

**Example 7.2.** If  $n > 4$  and digit sets is  $\mathcal{D} = \{0, 1, \dots, n - 2\}$ , then  $\mathcal{F} = [0, n - 2/n - 1]$  by Proposition 6.3. The Hausdorff and lower Minkowski dimensions of  $\mathcal{C} \cap (\mathcal{C} + x)$  are at least  $\log_n(n - 3)$  for any  $x = 0.x_1x_2 \dots$  such that  $x_1 < n - 2$ .

*Proof.* Since the separation condition  $d_{j+1} - d_j \geq 2$  is not satisfied we need a variant of the analysis in Section 2. Fix  $x = 0.x_1x_2 \dots$ . Let  $B_k$  be the number of intervals in  $\mathcal{C}_k$  that are both in the interval case and in the potential interval case and let  $I_k$  be the number of intervals in  $\mathcal{C}_k$  that are in the interval case but not in the potential interval case. We will ignore the intervals in  $\mathcal{C}_k$  that are in the potential interval case but not in the interval case.

Note that  $B_0 = 0$  and  $I_0 = 1$ .



FIGURE 7.1. Simultaneous interval and potential interval case, when  $n = 8$  and  $x_{k+1} = 3$

The reader may verify that for any digit  $x_{k+1} = 0, 1, \dots, n - 1$  we have

$$(7.1) \quad B_{k+1} \geq (n - 3)B_k, \text{ for } k \geq 0.$$

See Figure 7.1 for the case  $n = 8$  and  $x_{k+1} = 3$ . The bottom line illustrates an interval in  $\mathcal{C}_k$  and its refinement, the boldface line indicates the intervals retained after refinement. The top line illustrates the two intervals in  $\mathcal{C}_k + \lfloor x \rfloor_k$ .

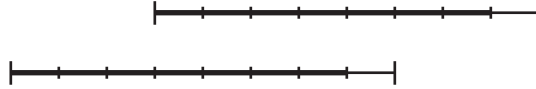


FIGURE 7.2. Interval case, when  $n = 8$  and  $x_{k+1} = 3$

Also, as illustrated in Figure 7.2

$$x_{k+1} < n - 2 \implies B_{k+1} \geq I_k, \text{ for } k \geq 0.$$

Consequently, if  $x_1 < n - 2$ , then

$$(7.2) \quad B_k \geq (n - 3)^{k-1}, \text{ for } k \geq 1.$$

If  $I$  is an interval in  $\mathcal{C}_k$  that is both interval case and the potential interval case, then the refinement of  $I$  contains an interval of the same type in  $\mathcal{C}_{k+1}$ . Hence, by (7.1) and the Nested Interval Theorem, any interval in  $\mathcal{C}_k$ , that is both in the interval case and in the potential interval case, contains infinitely many points from  $\mathcal{C} \cap (\mathcal{C} + x)$ . Consequently,

$$(7.3) \quad \mathcal{N}_k(\mathcal{C} \cap (\mathcal{C} + x)) \geq B_k$$

for all  $k \geq 1$ .

Suppose  $x_1 < n - 2$ . Combining (7.2) and (7.3) it follows that the lower Minkowski dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is at least  $\log_n(n - 3)$ .

Let  $\mathcal{A}_k$  be the intervals in  $\mathcal{C}_k$  that are both in the interval and the potential interval case. Above we showed the lower Minkowski dimension of  $\mathcal{B}_x = \bigcap_{k=1}^{\infty} \bigcup_{I \in \mathcal{A}_k} I$  is bounded below by  $\log_n(n - 3)$ . It follows from the argument in Sub-Section 3.2 that the Hausdorff dimension of  $\mathcal{B}_x$  is also bounded below by  $\log_n(n - 3)$ . Since  $\mathcal{B}_x \subseteq \mathcal{C} \cap (\mathcal{C} + x)$  the Hausdorff dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is also bounded below by  $\log_n(n - 3)$ .  $\square$

*Remark 7.3.* If  $n = 3$  in Example 7.2 then  $\mathcal{C}$  is the standard middle thirds Cantor set scaled by the factor  $1/2$ . Hence the conclusions of Theorem 1.1 hold in this case. It remains to consider the case  $n = 4$ . In this case an argument similar to the one given for Example 7.2, but also incorporating ideas from the argument given for Example 7.4, shows the conclusions of Theorem 1.1 fail.

The following example  $\mathcal{D}$  satisfies the separation condition  $d_{j+1} - d_j \geq 2$ , but  $d_m = n - 1$ .

**Example 7.4.** Let  $n = 8$ . If  $\mathcal{D} = \{0, 2, 4, 6\}$ , then we can apply Theorem 1.1. If  $\mathcal{D} = \{0, 2, 4, 7\}$ , then we cannot apply Theorem 1.1. By Proposition 6.3  $\mathcal{F} = [0, 1]$ . We claim the Hausdorff and lower Minkowski dimensions of  $\mathcal{C} \cap (\mathcal{C} + x)$  are least  $\log_8 \sqrt{2} = 1/6$ , for any  $x = 0.x_1x_2 \dots$  such that  $x_1 = 3$  or  $x_1 = 4$ .

*Proof.* Since the separation condition  $d_{j+1} - d_j \geq 2$  is satisfied we can use the analysis in Section 2. Fix  $x = 0.x_1x_2 \dots$ . Let  $I_k$  be the number of intervals in  $\mathcal{C}_k$  that are in the interval case and let  $P_k$  be the number of intervals in  $\mathcal{C}_k$  that are in the potential interval case.



FIGURE 7.3. Interval case with  $x_{k+1} = 0$

We claim that for any  $k \geq 0$  at least one of the following four sets of inequalities holds

$$\begin{array}{lll} I_{k+1} \geq 2I_k & \text{and} & P_{k+1} \geq P_k \\ I_{k+1} \geq I_k & \text{and} & P_{k+1} \geq 2P_k \\ I_{k+1} \geq 2P_k & \text{and} & P_{k+1} \geq I_k \\ I_{k+1} \geq P_k & \text{and} & P_{k+1} \geq 2I_k \end{array}$$

When  $x_{k+1} = 0$  this is illustrated in Figure 7.3 and Figure 7.4. The cases where  $x_{k+1}$  is one of 1, 2, 5, 6, 7 can be handled in the same way.

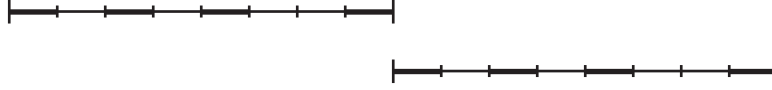


FIGURE 7.4. Potential interval case with  $x_{k+1} = 0$

When  $x_{k+1}$  is 3 or 4, illustrations similar to Figure 7.3 and Figure 7.4 show that  $I_{k+1} = I_k + P_k$  and  $P_{k+1} = I_k + P_k$ . If  $I_{k+1} \leq P_{k+1}$  the equalities imply, for example,  $I_{k+1} \geq 2I_k$  and  $P_{k+1} \geq P_k$ . A similar argument applies if  $P_{k+1} < I_{k+1}$ .

Assume  $x_1 = 3$  or  $x_1 = 4$ , then  $I_1 = P_1 = 1$ . Since at each stage we multiply one of  $I_k$  and  $P_k$  by at least 2 and the other by at least 1, it follows that

$$I_k \geq 2^{r_k} \text{ and } P_k \geq 2^{k-r_k-1} \text{ for } k \geq 1$$

where  $r_k$  are integers satisfying  $r_1 = 0$  and  $r_k \leq r_{k+1} \leq 1 + r_k$ . Since either  $r_k \geq (k-1)/2$  or  $k - r_k - 1 \geq (k-1)/2$  we have

$$\max\{I_k, P_k\} \geq 2^{(k-1)/2}$$

when  $k \geq 1$ . By an argument in Example 7.2

$$\mathcal{N}_k(\mathcal{C} \cap (\mathcal{C} + x)) \geq \max\{I_k, P_k\}.$$

Consequently, the lower Minkowski dimension of  $\mathcal{C} \cap (\mathcal{C} + x)$  is at least  $\log_3 2^{1/2}$ . As in Example 7.2 the argument from Sub-Section 3.2 implies that the Hausdorff dimension has the same lower bound.  $\square$

## 8. CONCLUDING REMARKS

Suppose  $n = 3$  and  $\mathcal{D} = \{0, 2\}$ . In Section 3 we constructed  $t = 0.t_1 t_2 \dots$  that do not admit a finite ternary representation such that  $\mathcal{C} \cap (\mathcal{C} + t)$  has Hausdorff dimension  $\log_3 2$ . It follows from a result due to Nekka and Li [5] that the  $\log_3 2$  dimensional Hausdorff measure of  $\mathcal{C} \cap (\mathcal{C} + t)$  equals zero. Hence, by Hutchinson's theorem [3]  $\mathcal{C} \cap (\mathcal{C} + t)$  cannot be a self-similar set satisfying the open set condition. So for a deleted digits Cantor set  $\mathcal{C}$  an interesting problem is to characterize the  $t$ 's for which  $\mathcal{C} \cap (\mathcal{C} + t)$  is self-similar.

We showed that  $\mathcal{G} = -\mathcal{F} \cup \mathcal{F}$  is self-similar and if the digit set  $\mathcal{D}$  is a subset of an arithmetic progression, then  $\mathcal{G}$  is similar to a deleted digits Cantor set. Does there exist  $n$  and  $\mathcal{D}$  such that  $\mathcal{F}$  is not an interval, yet  $\mathcal{F}$  is self-similar or even a deleted digits Cantor set  $\mathcal{F} = \mathcal{C}_{m,\varepsilon}$ ?

The examples in Section 7 suggests that, when the assumptions in Theorem's 1.1 and 1.2 fail, it may still be possible to find  $a$ , perhaps in terms of  $0.t_1 t_2 \dots t_n$ ,

such that for any  $a \leq \alpha \leq 1$ , there is a “completion”  $t = 0.t_1t_2 \cdots t_nt_{n+1} \cdots$  for which  $\mathcal{C} \cap (\mathcal{C} + t)$  has dimension  $\alpha \log_n m$ .

## REFERENCES

- [1] Besicovitch, Abram S., On existence of subsets of finite measure of sets of infinite measure, *Indagationes Math.* **14** (1952), 339–344.
- [2] Falconer, Kenneth J., *The geometry of fractal sets*, Cambridge University Press, Cambridge, 1985.
- [3] Hutchinson, John E., Fractals and self-similarity, *Indiana Univ. Math. J.* **30** (1981), 713–747.
- [4] Davis, Gregory J. and Hu, Tian-You, On the structure of the intersection of two middle third Cantor sets, *Publ. Mat.* **39** (1995), 43–60.
- [5] Nekka, Fahima and Li, Jun, Intersections of triadic Cantor sets with their translates - I. Fundamental properties, *Chaos Solitons Fractals* **13** (2002), 1807–1817.
- [6] Dai, Meifeng and Tian, Lixin, On the intersection of an  $m$ -part uniform Cantor set with its rational translation, *Chaos Solitons Fractals* **38** (2008), 962–969.
- [7] Li, Jun and Nekka, Fahima, Intersections of triadic Cantor sets with their translates, II. Hausdorff measure spectrum function and its introduction for the classification of Cantor sets, *Chaos Solitons Fractals* **19** (2004), 35–46.
- [8] Eggleston, H. G., The fractional dimension of a set defined by decimal properties, *Quart. J. Math., Oxford Ser.* **20** (1949), 31–36.

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